Section 15.1:
Double Integrals Over Rectangles

What We'll Learn In Section 15.2

1. The Integral (Calc. 1 Version, $f(x) \geq 0$, area)
2. The Double Integral (Over a Rectangle, $f(x, y) \geq 0$, volume)
3. Iterated Integrals
4. General Integrals

1. The Integral (Calc. 1 Version)

Let $f(x)$ be a continuous function on [a, b]. ($f(x) \geq 0$, finding area under curve case)

n = # of subintervals = # of rectangles

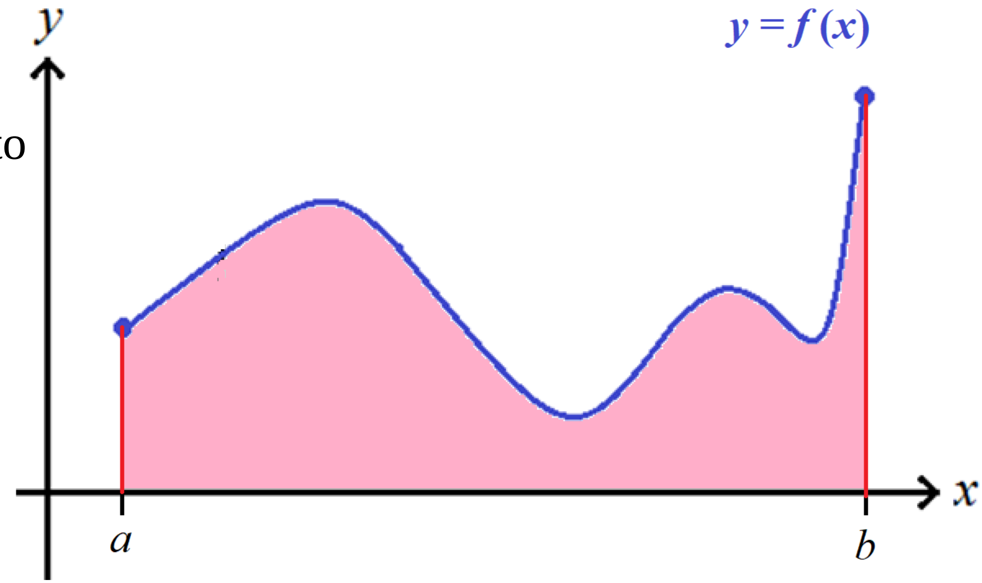
$x_0, x_1, x_2, \dots, x_n$ = the endpoints of the intervals

$\Delta x = \frac{b-a}{n}$ = the width of each subinterval

$x_1^*, x_2^*, \dots, x_n^*$ = the point in each interval used to determine the height of each rectangle

$f(x_1^*), f(x_2^*), \dots, f(x_n^*)$ = the height of each rectangle

$f(x_1^*)\Delta x,\ f(x_2^*)\Delta x, \dots, f(x_n^*)\Delta x$ = the area of each rectangle



The ith subinterval = $[x_{i-1}, x_i]$

The point chosen in the ith subinterval = x_i^*

The height of the ith rectangle = $f(x_i^*)$

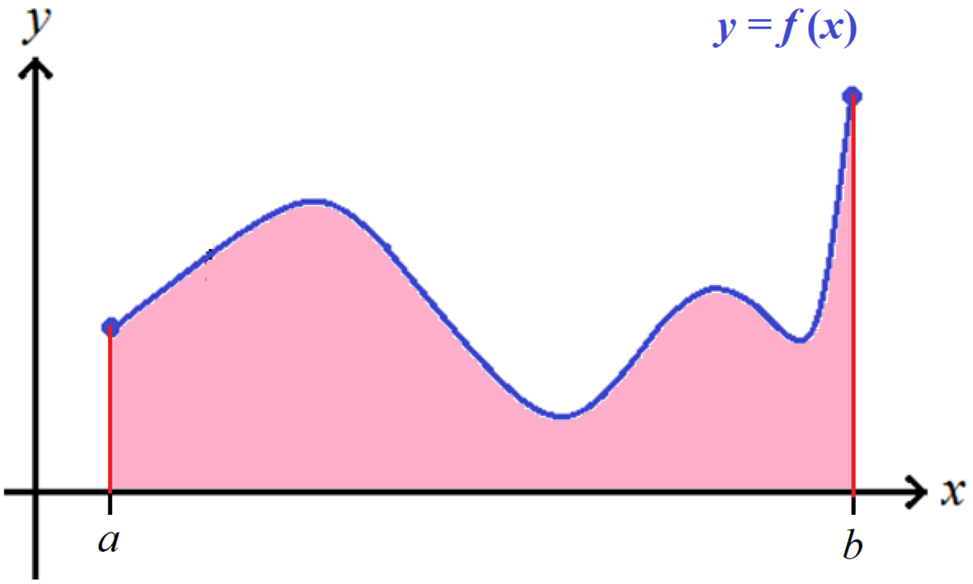
The area of the ith rectangle = $f(x_i^*)\ \Delta x$

$A_n = \sum_{i=1}^{n} f(x_i^*)\Delta x$ = the total area of all of the rectangles

1. The Integral (Calc. 1 Version)

Let $f(x)$ be a continuous function on [a, b]. ($f(x) \geq 0$, finding area under curve case)

$A_n = \sum_{i=1}^{n} f(x_i^*)\Delta x =$ the total area of all of the rectangles



A_n is a sequence

A_n is convergent and always converges to the same answer no matter what points in each interval were chosen

Notation for this limit is…

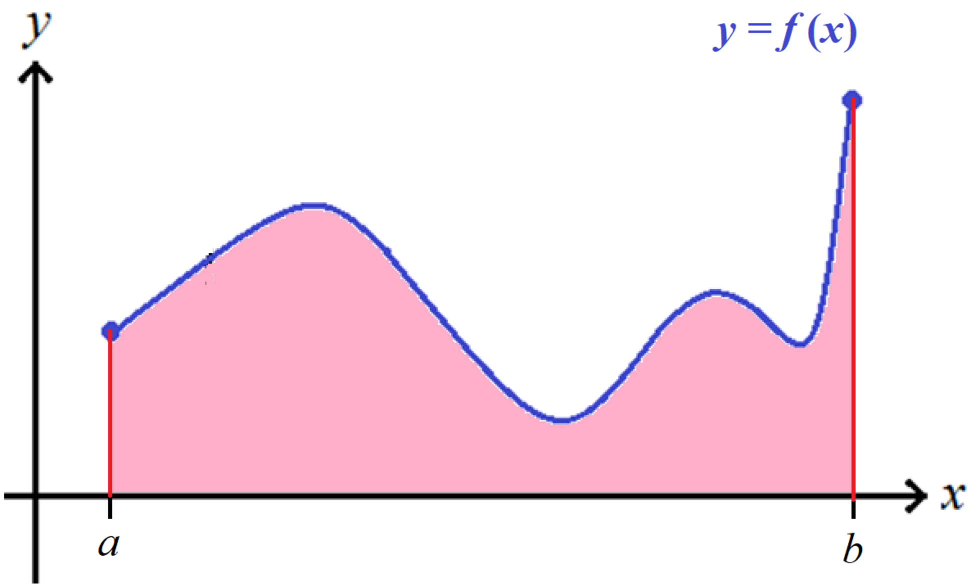
$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx$$

1. The Integral (Calc. 1 Version)

Let $f(x)$ be a continuous function on [a, b]. ($f(x) \geq 0$, finding area under curve case)

A = area we're looking for

If in each interval, you choose the point where the graph is the lowest, you get a sequence L_n and $L_n \leq A$



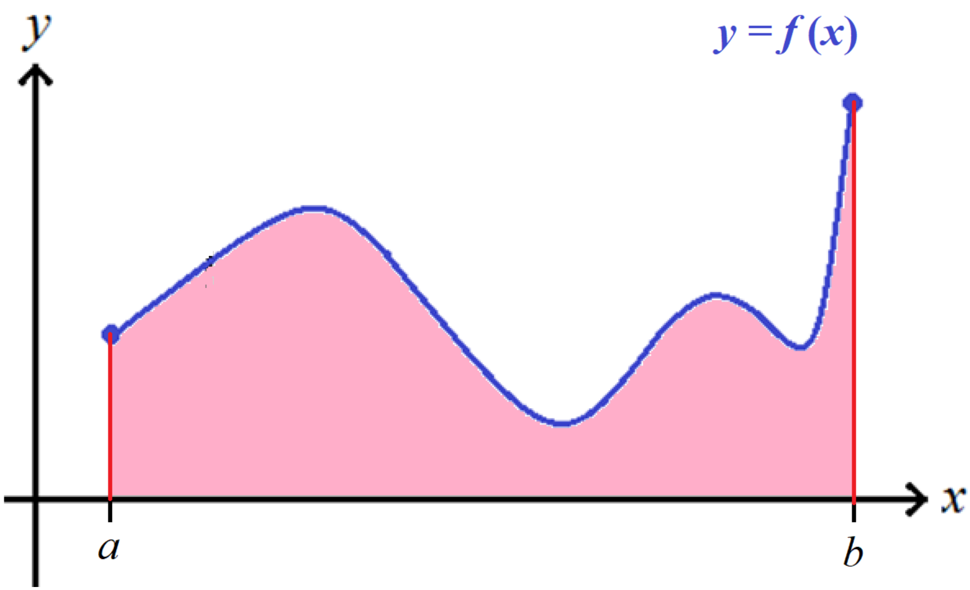
If in each interval, you choose the point where the graph is the highest, you get a sequence U_n and $U_n \geq A$

1. The Integral (Calc. 1 Version)

Let $f(x)$ be a continuous function on [a, b]. ($f(x) \geq 0$, finding area under curve case)

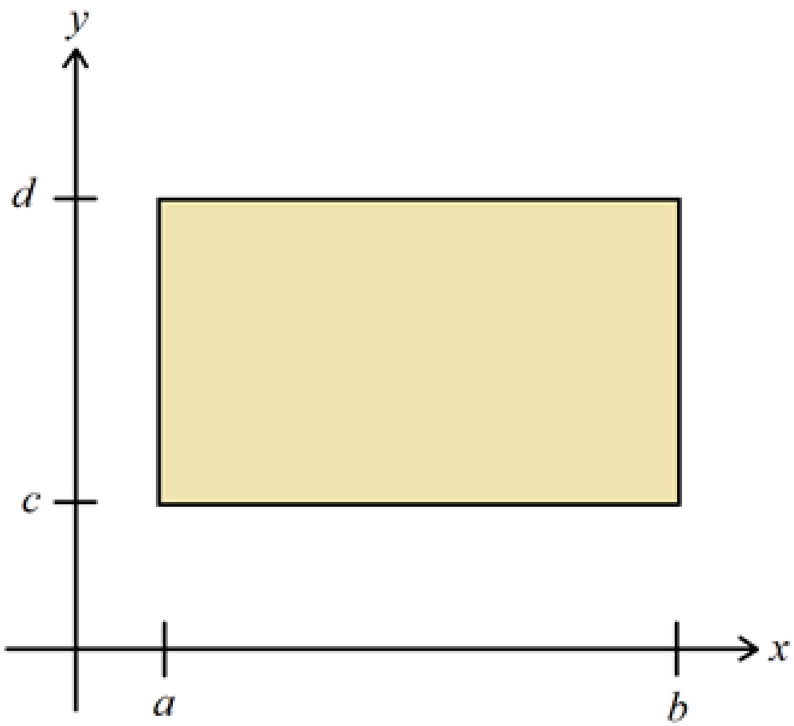
So…

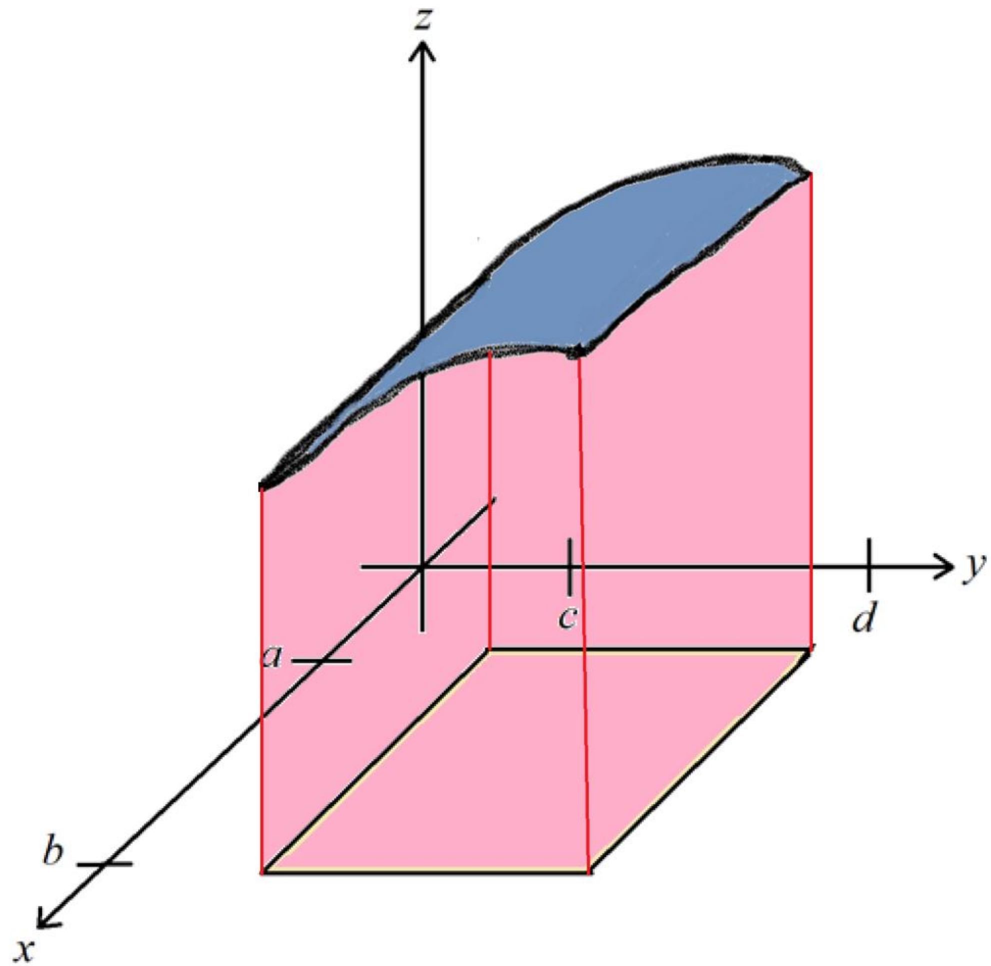
$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n = \lim_{n\to\infty} A_n = \int_a^b f(x)\, dx$$



2. The Double Integral (Over a Rectangle)

Let $f(x, y)$ be a continuous function on [a, b]×[c, d].
($f(x, y) \geq 0$, finding volume under surface case)





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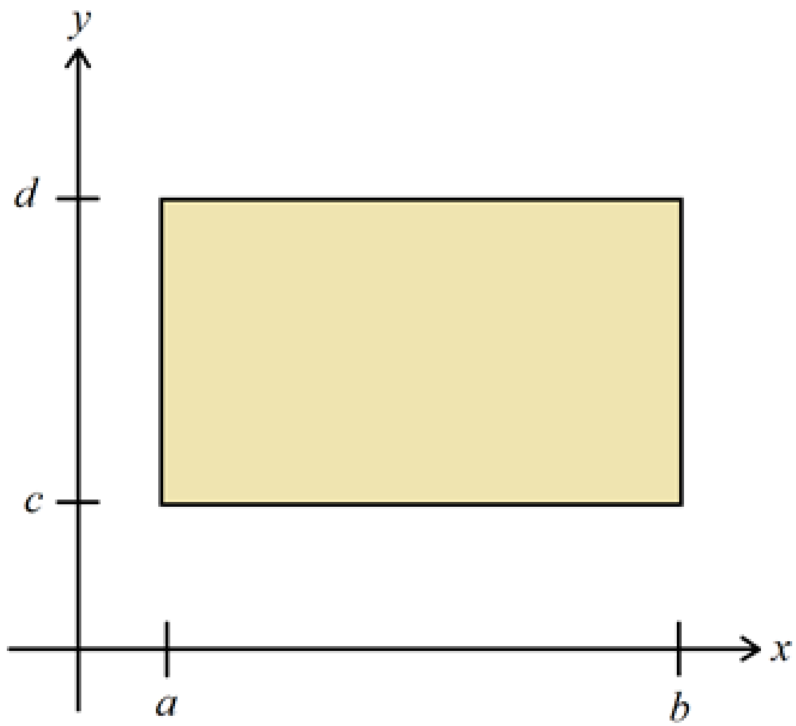

Divide [a, b] into m subintervals

Divide [c, d] into n subintervals

This divides the rectangle into subrectangles

The lengths and widths of each subrectangle are Δx and Δy

The selected point in each sub-rectangle is (x_{ij}^*, y_{ij}^*)

The height of each rectangular box is $f(x_{ij}^*, y_{ij}^*)$

The volume of each rectangular box is $f(x_{ij}^*, y_{ij}^*)\Delta A$ or $f(x_{ij}^*, y_{ij}^*)\Delta x \Delta y$

The total volume of all of the rectangular boxes is $$A_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$$

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$A_{m,n}$ is a (doubly indexed) sequence

$A_{m,n}$ is convergent and always converges to the same answer no matter what point is picked in each subrectangle

$$V = \lim_{m,n\to\infty} A_{m,n} = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^{*}, y_{ij}^{*}\right) \Delta A = \iint_{R} f(x, y)\, dA$$

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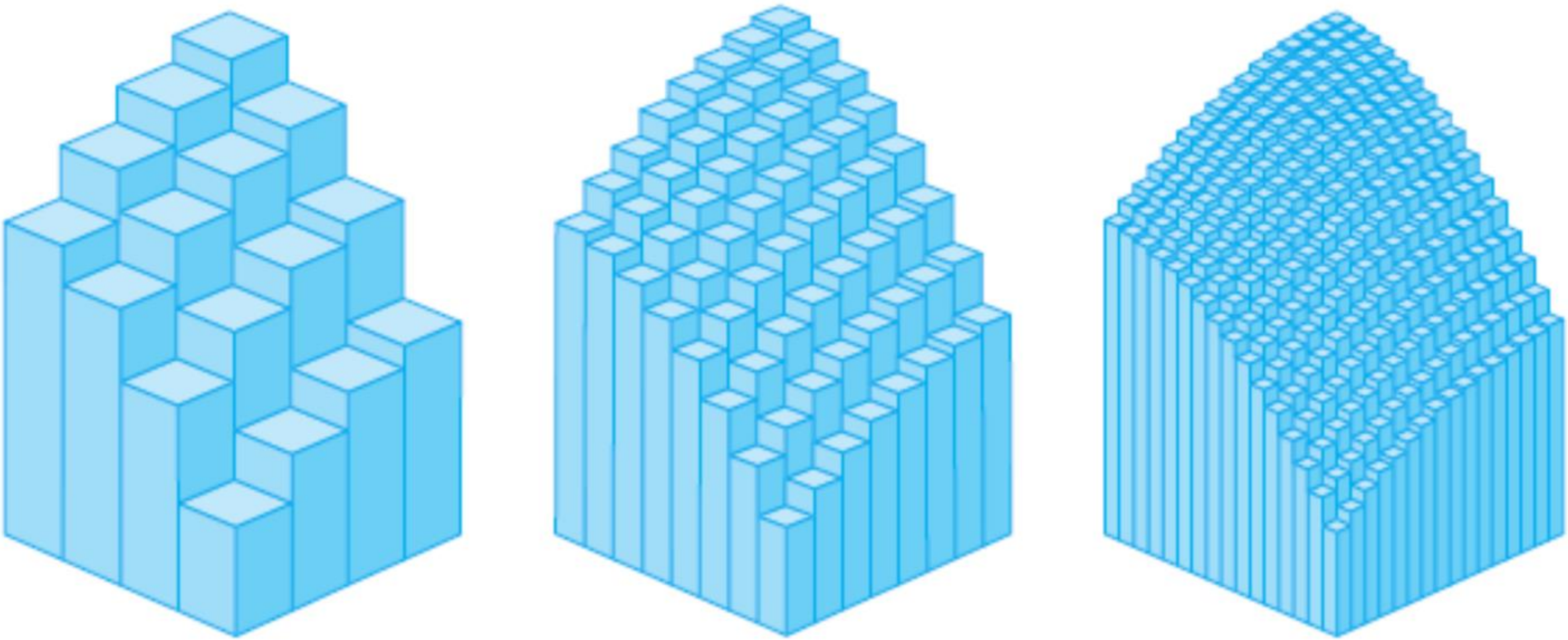

(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

2. The Double Integral (Over a Rectangle)

Ex 1: Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes.

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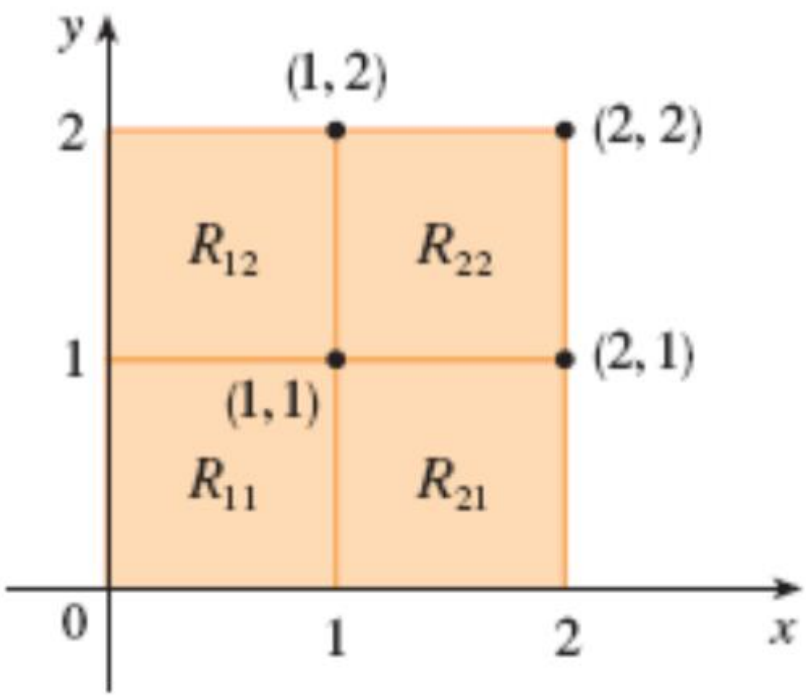

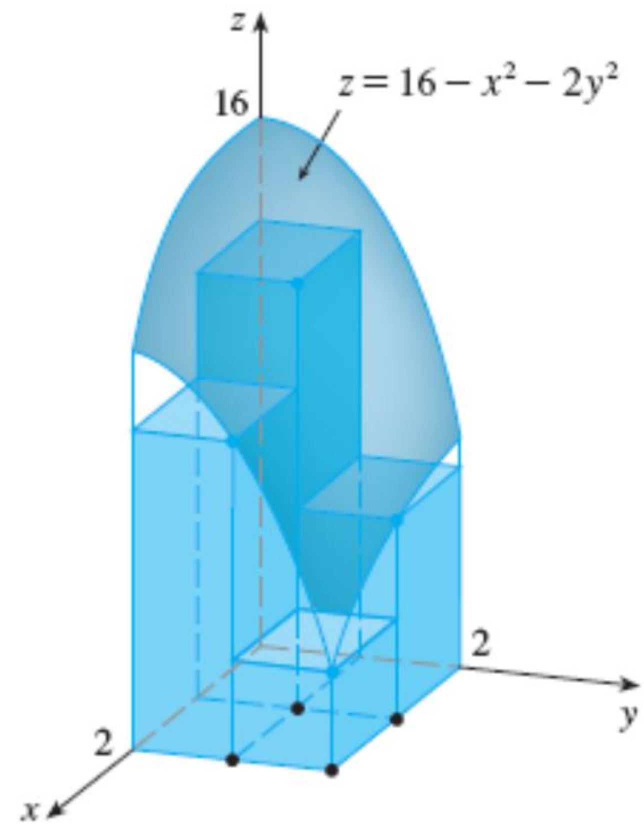

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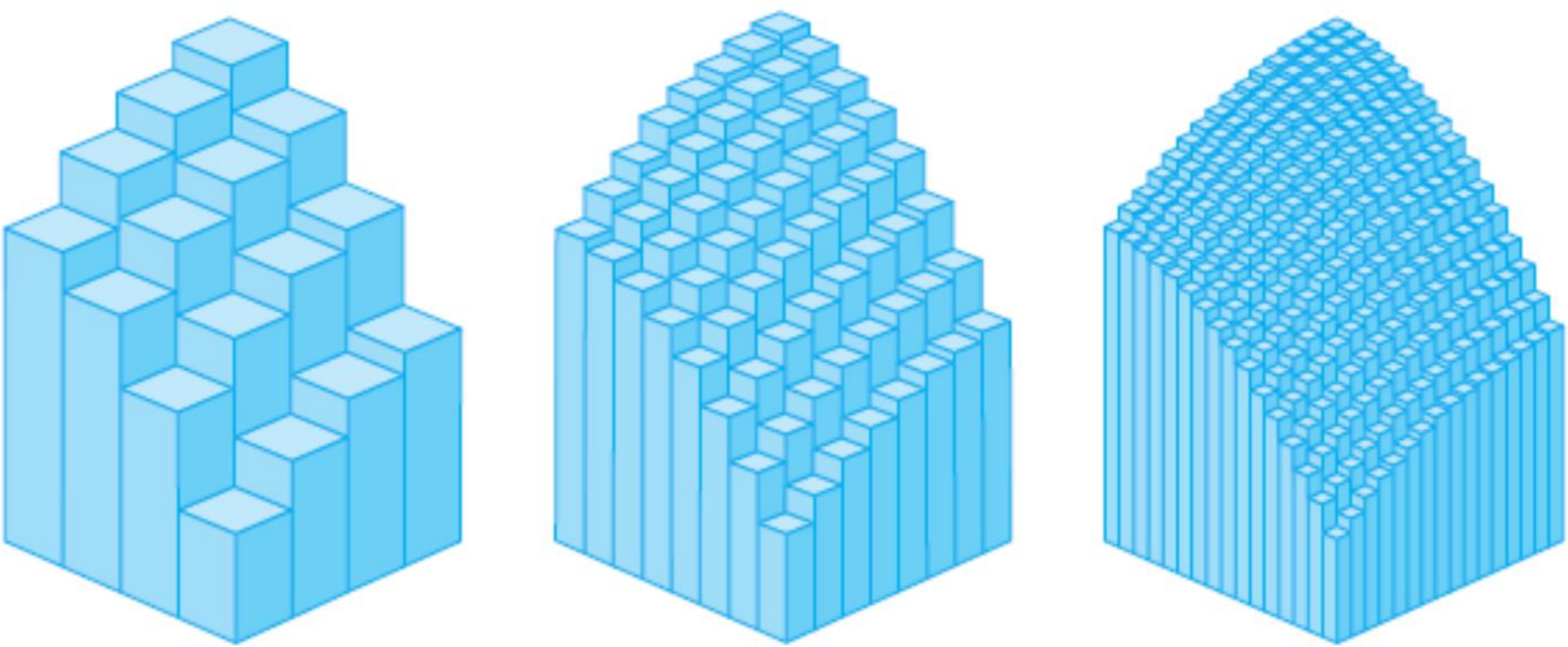

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2. The Double Integral (Over a Rectangle)

Ex 2: If $R = \{\, (x, y) \mid -1 \leq x \leq 1\, , -2 \leq \mathrm{y} \leq 2\,\}$, evaluate the integral

$$\iint_R \sqrt{1 - x^2}\, dA$$

2. The Double Integral (Over a Rectangle)

Midpoint Rule for Double Integrals

$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(\overline{x}_i, \overline{y}_j\right)\, \Delta A$$

where $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

2. The Double Integral (Over a Rectangle)

Ex 3: Use the midpoint Rule with $m = n = 2$ to estimate the value of the integral below, where $R = \{\,(x, y)\mid 0 \le x \le 2\,,\ 1 \le \mathrm{y} \le 2\,\}$,

$$\iint\limits_R (x - 3y^2)\, dA$$

3. Iterated Integrals

Ex 4: Evaluate the iterated integrals below:

a) $\int_0^3 \int_1^2 x^2 y \, dy \, dx$

3. Iterated Integrals

Ex 4: Evaluate the iterated integrals below:

b) $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

3. Iterated Integrals

Fubini's Theorem

If f is continuous on the rectangle $R = \{(x, y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$, then

$$\iint\limits_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

More generally, this is true if we assume that f is bounded on R , f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

3. Iterated Integrals

Ex 5: Evaluate the double integral below in 2 different ways, where $R = \{\ (x, y) \mid 0 \le x \le 2\ ,\ \ 1 \le \mathrm{y} \le 2\ \}$.

$$\iint_R (x - 3y^2)\, dA$$

3. Iterated Integrals

Ex 6: If $R = [1, 2] \times [0, \pi]$, evaluate

$$\iint\limits_R y\sin(xy)\, dA$$

3. Iterated Integrals

Ex 7: Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

3. Iterated Integrals

$$\iint_R g(x)\,h(y)\;dA = \int_a^b g(x)\;dx \int_c^d h(y)\;dy$$

where $\qquad R = [a, b] \times [c, d]$

3. Iterated Integrals

Ex 8: If $R = [0, \pi/2] \times [0, \pi/2]$, evaluate

$$\iint_R \sin x \cos y \; dA$$

4. General Integrals

$\int_a^b f(x)\, dx$ can still be calculated even if $f(x)$ is not always ≥ 0.

It's still a limit of Riemann sums.

It just loses its meaning as an area under the curve $f(x)$

This kind of limit show up a lot, especially in physics…

4. General Integrals

Let $f(x)$ be a continuous function on [a, b]. ($f(x)$ not necessarily ≥ 0)

Divide the interval [a, b] into n subintervals.

In each subinterval $[x_{i-1}, x_i]$, pick a point x_i^*.

$f(x)$ $f(x_i^*)\,\Delta x$

Calculate $\displaystyle\sum_{i=1}^{n} f(x_i^*)\,\Delta x$

This sum approximates some quantity you're interested in

Then $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$

gives you the exact value of the quantity you're interested in

4. General Integrals

<u>Ex 9</u>: A rod is lying along the x-axis from $x = 0$ to $x = 4$ and has a linear density given by $\rho(x) = 9 + 2\sqrt{x}$ where x is measured in meters and ρ is measures in kilograms per meter. Find the total mass of the rod.

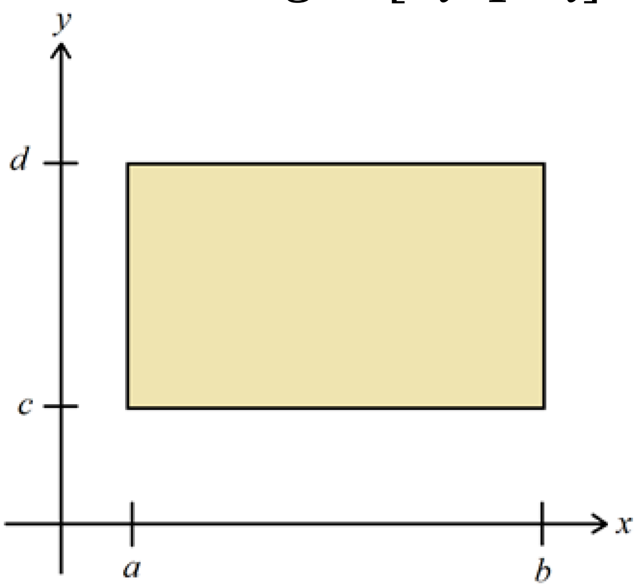
Idea…

4. General Integrals

Let $f(x, y)$ be a continuous function on $[a, b] \times [c, d]$.
$(f(x, y)$ not necessarily $\geq 0)$

Divide the interval [a, b] into m subintervals.
Divide the interval [c, d] into n subintervals.

In each rectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$, pick a point (x_{ij}^*, y_{ij}^*).



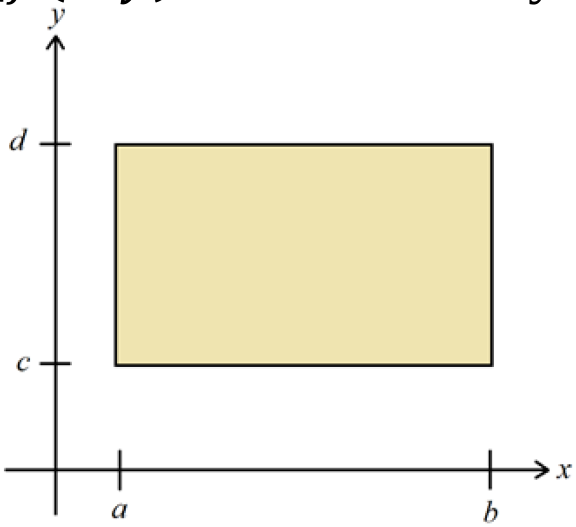
$f(x, y)$ $\qquad$ $f(x_{ij}^*, y_{ij}^*)\,\Delta A$

Calculate $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$

This sum approximates some quantity you're interested in

4. General Integrals

Let $f(x, y)$ be a continuous function on $[a, b] \times [c, d]$.
($f(x, y)$ not necessarily ≥ 0)



Calculate $\sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$

This sum approximates some quantity you're interested in

Then $\iint_R f(x, y)\, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$

gives you the exact value of the quantity you're interested in

4. General Integrals

Ex 10: A rectangular sheet of metal is lying in $[1,2] \times [1,4]$ region of the *xy*-plane. It has an areal density given by $\rho(x, y) = x^2 y + x + 1$ where x and y are measured in meters and ρ is measures in kilograms per square meter. Find the total mass of the sheet.

Idea…